Dialogs around Models and Uncertainty
Interview of Arthur Jaffe by Pauline Barrieu
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In one or two sentences, can you describe your field?
My field is to understand nature as mathematics. What that means is we try to understand the world through a logical (mathematical) description of physical phenomena, one that also has the power of prediction.

My own special interest has for many years focused on the two biggest achievements of 20^{th} century physics—relativity and quantum theory. One could regard these subjects as the elephants in the room. Finding each of them revolutionized our understanding of Nature, and physics has focused on them ever since their discovery. Relativity deals mainly with the macro-world of classical physics and the cosmos. Quantum theory deals mainly with the micro-world. It not only frames our understanding of elementary particles, atoms, and molecules, but it also provides the framework for these objects to be the building blocks of gases, liquids and solids—namely all of nature. Both relativity and quantum theory have had enormous practical importance in our life.

Yet after almost one hundred years of studying relativity and quantum theory separately, we still do not understand whether these two concepts are *mathematically* compatible with one another! This leads us to ask an even more fundamental question: is it possible to describe nature through mathematics and to understand both of these phenomena? The alternative is we can access Nature only through understanding different and imperfect models whose domain of relevance may not overlap.

In the general public, most persons take for granted that physics is an all-encompassing and fundamental science, actually part of mathematics. However in the field, experts harbour a lurking doubt whether we can find a model that both describes physical phenomena in detail, and also meets the standards of mathematics.

And can you give us an example of a typical model in your field?
One of the simplest and oldest models forms the starting point of most investigations in mathematical physics. It is the mathematical representation of space and time in the universe by a continuum of points. It provides the fundamental arena in which we describe our world. This picture permeates Euclidean geometry that we learn in school; it is the foundation of Newton's calculus; it forms the basis for Maxwell's theory of electromagnetism; and it remains the basis for modern developments—including relativity and quantum theory. We really do not know whether this space-time continuum is a correct model, yet it forms the basis for most of theoretical physics.

A more complicated model, based on this continuum, is to assume that space-time is compatible with relativity. For special relativity this means that space-time is the space described by Minkowski, which provides a model space-time that appears to allow for a description of elementary particles.

We'll come back to this a bit later, but would you say you use models mainly to describe Nature?
Absolutely. Models to describe nature are the basis for all my work. The idea of such a model is to attempt to isolate a part of physics in a self-contained way, so one can describe completely an idealization of nature which is a subset of the world. Such models underlie all of physics; by extension they also include models for chemistry, for engineering, or even for finance. And when one considers such models to describe Nature, they should be logically sound—even though they may be applicable only in special realms.

So models describe the state of Nature and also movements or evolution?

Evolution is a central feature of Nature, so evolution in time needs to play a central role in a model. Often the model arises as a mathematical equation that describes the time evolution of Nature. This is the case in quantum theory, where the equations of Schrödinger, Heisenberg, and Dirac play that role. These equations can come in many different forms, and finding the appropriate form of the equation comes down to discovering a "law of physics." This provides the model.

However such a dream is grandiose, for one must also show that the equation one conjectures makes sense as mathematics. In the case of planetary motion, Newton had to invent calculus to do that. For the problems with quantum theory and special relativity, one also needs to invent new mathematics in order show that the equations themselves are mathematically consistent. This is what people in physics try to do: they try to find the equations of evolution, and by doing that they discover the constituents of Nature. They make predictions based on the model. However the key conceptual point is that they also need to show that the model fits into mathematics: old or new.

And so with the help of this example, could you explain why a model is needed, and what it is?
A model is needed because we want to translate the world into a set of mathematical statements or equations. In order to describe something by an equation mathematically you need to have some idea of what the symbols mean. And the model provides the idea of what the symbols mean. Thus the model may contain the notion of "particles" as something that one can derive from the equations. And forces between particles should also arise as a consequence of finding the right equations (model).

So my next question, which is what is the role of mathematics, I think it's quite essential from what you are saying?
Everything in theoretical physics revolves around mathematics. This really means that one has a logical framework. Mathematics isn't fixed – mathematics keeps changing as one discovers new ideas. And by inventing a new model, it may be necessary to invent new mathematics in order to understand it. And that's been the history of this subject – that by understanding Nature we actually discover new mathematics. For example, understanding quantum theory without relativity led to many new insights into practical problems about the real world. But it also led to new insights and theories about differential equations, about analysis, about probability theory, about algebra, about representation theory, and about geometry. The new model of quantum theory also revolutionized mathematics!

So is it like inventing new tools to describe something? Or - it's more than tools, isn't it?
Yes. You can think of starting with certain primitive tools, but then refining your tools as you use them. There is a back-and-forth between the discovery of the model, the development of the tools, and the understanding of the model. The tools can be used to understand things, but then have a life of their own – they have their own families. They have their children and they go on merrily; but sometimes these children turn out to be related to things back in the real world that you never imagined when starting out!

So a naïve question: would you say that sometimes mathematics, this new mathematics, will go further than what they are needed to describe or to represent, and then it makes you understand new things from Nature?
Absolutely. And in fact historically, when quantum theory was invented, the modern quantum theory started in the 1920s with the work of Jordan and Heisenberg, and also Dirac, but the mathematics associated with that became so complicated that people in the 1950s began to think that maybe it was not even possible to describe Nature by mathematics. Because the theory appeared so complicated, they came to believe that it was impossible to get to the end. And so the mathematics which had been tied to physics historically in the last century seemed to divorce from physics. And now we're trying to get back together again.
I like your way of presenting these things – people might be surprised.

Once new tools were discovered, it turned out that the tools themselves have a much wider validity. The tools can be used in engineering, in economics, in medicine and so forth—in ways unimaginable at the time the tools were invented. This is an old story, which I wrote about some time ago in an essay "Ordering the Universe: the Role of Mathematics". Let me quote a bit from the essay about Fourier analysis:

"In the early 1800s, Jean Baptiste Joseph Fourier, newly returned from his post as civil governor of Napoleonic Egypt, set out to understand the problem of heat conduction. Given the initial temperature at all points of a region, he asked, how will heat diffuse over the course of time? It was curiosity about such phenomena as atmospheric temperature and climate that led Fourier to pose the abstract question. In order to solve the heat diffusion equation, Fourier devised a simple-but brilliant-mathematical technique. This equation turned out to be easy to solve if the initial heat distribution were oscillatory-that is, essentially a sine wave. To take advantage of this, Fourier proposed decomposing any initial heat distribution into a (possibly infinite) sum of sine waves and then solving each of these simpler problems. The solution to the general problem could then be obtained by adding up the solutions for each of the oscillatory components, called harmonics.

"French mathematicians, such as Lagrange, sharply rejected the idea, doubting that these simple harmonics could adequately express all possible functions, and casting aspersions upon Fourier's rigor. These attacks dogged Fourier for two decades, during which he carried his research forward with remarkable insight. Today we owe an enormous debt to his remarkable tenacity, his stubbornness, and his ability to proceed in spite of formidable doubts in the minds of the leaders of the scientific establishment. Fourier found it difficult to publish his work even after he received the 1811 grand prize in mathematics from the Académie des Sciences for his essay on the problem of heat conduction, because the academy's announcement of the award expressed grave reservations concerning the generality and rigor of Fourier's method. Fourier persevered and finally his work won general acceptance with the publication of his now-classic *The Analytic Theory of Heat*, in 1822.

"The method of harmonic analysis, or Fourier analysis, has turned out to be tremendously important in virtually every area of mathematics and physical science, much more important than the solution of the problem of heat diffusion. In mathematics, it has become a subject by itself. But in addition the theories of differential equations, of group theory, of probability, of statistics, of geometry, of number theory, to mention a few, all use Fourier's technique for decomposing functions into their fundamental frequencies. In physics, engineering, and computer science the effect has been no less profound. At least as important as the numerous applications to science and engineering has been the application of Fourier analysis to mathematics itself. Like other scientists, mathematicians are constantly searching for new tools to solve their theoretical problems. Frequently it happens that techniques discovered to solve one abstract problem later apply to a wide variety of others.

"If you need to be convinced of this, look under "Fourier" in the catalogue of a university science library. At Harvard's, for example, there are 212 entries, of which the first ten are Fourier analysis in probability theory, Fourier analysis in several complex variables, Fourier analysis of time series, Fourier analysis of unbounded measures on locally compact abelian groups, Fourier analysis on groups and partial wave analysis, Fourier analysis of local fields, Fourier analysis of matrix spaces, Fourier coefficients of automorphic forms, the Fourier integral and its applications, and Fourier integral operators and partial differential equations."

So beside this mathematical world, what do you have in a model? Would you say that you have something else besides mathematics?

If you want the model to reflect the world around us you have to have the concepts that we see in the world around us as part of the model. So I gave you a very simple example before – points in space-time – but you want to be able to construct objects, and other things coming out of your picture, your model. So in fact the model that I was talking about, the consistency of relativity and quantum theory, these are based on descriptions where we would like to predict particles that exist and have been seen at CERN and in other places in laboratories, and you would like to predict the forces between them, based on some simple principles

Are these principles the results of some other models? Or are they based on some evidence?
The history of models about Nature has to be hierarchical; it needs either to include what came before, or else a new model needs to incorporate earlier success of models in the same area. One should be able to reproduce what has been understood in the past.

What is the role of language in modelling? Does it have any role at all?
Everything has to be framed in language. And in fact mathematics is the language of the models I use. It's hard to separate the two. And when you get into these questions very deeply you're forced immediately into questions of philosophy, and so it becomes quite complicated. It is true that sometimes complicated or catchy names bring attention or notoriety to a particular model. But this "marketing" is something I try to avoid.

So do you have any more qualitative aspects in your modelling? I guess like the description of what you mentioned, these particles – I guess there is a descriptive part, a bit more qualitative?
Well there was a famous lecture given in the 1960's by Mark Kac. It arose from the coincidence of geometry and analysis. The theme was what you could learn from knowing the frequencies you hear from a drum. The question is whether you could actually tell the shape of the instrument by listening to it carefully, so Kac asked, 'Can you hear the shape of a drum?' This became a very famous question, because of course you can find out certain things from hearing certain tones coming out of the drum, but people wondered whether you describe the shape of the drum completely. It turns out that the answer is no. But it took a long time to find that out. You can find the size, you can find the perimeter, and you can find out many things about the drum; but you can't completely determine its shape. And now we have, coming out of physics, another mathematician, Alain Connes, who invented a new type of "non-commutative" geometry. We don't know if it really applies to Nature, but certainly it would be beautiful to build quantum theory into geometry. In any case, he asked, "Can you hear non-commutative geometry?" While we don't yet have the answer to that, these qualitative questions linger. Such theoretical questions go beyond the more detailed things that a scientist might measure in a laboratory; they ask about what you *are able to* find out, rather than details of what you *actually* observe.

How important is notation in modelling?
Notation can be very important, both in its aid to identifying and relating to the model, and also in emphasizing and encapsulating the simplicity of a model.

Coming back to what models are used for: models are often said to represent a target system, some aspect of the world. Does this characterisation describe what happens in your field?
Well yes, and this is actually part of the big question because historically in my field people tried to isolate a small part of Nature and to describe it mathematically. This goes back to Newton's description of planetary motion, or to Maxwell's description of electro-magnetism, or to Boltzmann's description of statistical physics, or Gibbs' description of statistical physics, and Einstein's description of relativity, or Schrödinger and Heisenberg's description of quantum theory. They all isolate some small part or phenomena that can be described very well. But when you try to put these things together the situation gets much more complicated. And we don't know, for instance, if electro-magnetism interacting with matter can be described completely mathematically within quantum theory. This is really a big question in physics. It is the question of how within that field a small part of Nature can be isolated and described. For that is the basis of a model.

In fact we think that probably the answer to the question about electromagnetism is "no." And that drives people a little bit crazy, because they think they have to include more and more and more. But then the theory gets more and more complicated and the model gets more and more complicated, and the understanding of it mathematically also gets more complicated. So in the end we don't know if you can make it all work!

So would you say somehow it's easier to have a global model, but if it becomes really complicated then you have to slice or focus on certain aspects of the world?
No what I'm trying to say is, if you have a global model it becomes very difficult to understand how it all fits together, to understand the whole thing, in fact it challenges the human mind to try and do it. And if you don't have a global model then even simple things that you want to put together don't seem to fit. So you're in a conundrum.

So it's a bit like climate and weather?
Yes, it's like climate and weather, where you believe you know the equations but you can't solve them; and you model them on a computer but you don't really trust the solutions. You don't know how much data to put in to predict the future.

We will come back to this a bit later. How do you understand the model-world interface?
Oh I don't know that I understand that really well; that's the experimental part of science and I'm not an expert on that at all, so I really can't comment very much about it in detail. But I do know that the most accurate measurements that are made in the world have to do with the 13 decimal point accuracy to which one has measured the magnetic moment of the electron—namely the reaction of the electron when one puts it in a magnetic field. And this experiment agrees down to the last decimal place with rules that physicists devised to calculate the number. In fact these rules are extremely complicated and took some sixty years to refine and work out. Yet we do not know that these incredible triumphs of experimental and theoretical science can be backed up by a mathematical theory like those theories of classical physics that we know from history.

What is the relationship between a model and theory?
Well, the model is what you have to base your theory on. Models have to be simple, elegant, beautiful, in order for them to be appealing. But once they're appealing you don't know if they really fit together and work, and so that's where the mathematics comes in, you have to turn the model into mathematics, and then it becomes a theory.

So you would say a model is more a relationship between different constituents, but it can be a sort of logical relationship?
I would say a model is a hypothesis, and the theory is like a mathematical proof.

So the model is like a conjecture, and then…?
It's a conjecture, and a theory is some proof that it fits together and maybe also that it applies to the world.

How do people come up with the conjectures? Is it based on existing theory and somehow you go back up to this upper level of conjecture? How do people come up with the conjecture?
You have to be…it takes a certain genius. You have to have bright eyes, and some nice ideas, and then you have to take into account experience and where you're aiming, that's the way you do it.

And then the theory: the theory proves that the conjecture is right, that it's actually working? It leads to another conjecture I would say?
Of course, yes. But let me tell you a little story: when I was a student I heard a lecture by Paul Dirac. He was very important in the early development of quantum theory, and he gave a lecture in Princeton. He talked about the history of the relativistic equation for the electron, which most people call the "Dirac equation," although he never used his own name in speaking about things. In any case, Dirac gave this beautiful lecture at the Institute for Advanced Study, in the seminar of Robert Oppenheimer, who said: 'Professor Dirac, at what point did you realize that your equation

was correct?' And Dirac replied (I'll say it in the first person), 'I was working at home and had to go to the library in order to look up the spectrum of hydrogen. But, as I was getting my bicycle out of the garage, I realized that this equation was so beautiful it had to be right!'
Ah! I think we will come back to this when we have the question about what is a good model, because this is so beautiful as an answer! It's very interesting, and quite different from what other experts in other fields have said about models.

What is the aim and use of a model? Here I have mentioned a few things, but it might be completely different. So learning, exploration, optimization, exploitation? It really depends I guess on the field. In your field what would you say?
I think the use of the model is to have something specific to focus on, and then to pose a question that might have a concrete answer. With a model one does not need to focus on Hilbert's question 'Can one axiomatize physics?' Rather one can ask whether a particular model of a part of physics is relevant.

So the aim is really to have a sort of representation or a better understanding of how things work?
Yes, to have some conjecture of how things work.

Do you use any computer simulations – or are computer simulations used in your field, not necessarily by you, but by others?
Generally I don't use computer simulations in my own work. Occasionally I use them to test some idea, in order to see if it could possibly be right. In this case the computer simulation gives a first idea, and then you try, based on the outcome of the simulation, to actually prove it mathematically. My own use of simulations is very elementary and limited.

But other people use computer simulations to test ideas, and some people even use computer simulations to try to formulate new ideas, or to actually find approximate solutions to equations. Some people even use computers to test mathematical theories: in geometry or in number theory. In the latter one has used a computer to locate many millions of zeros of the Riemann zeta function. The answers all fit into the conjecture of Riemann; but while this leads to some degree of confidence, we do not know that this (possibly the most famous unsolved) mathematical conjecture is true. And only with a true mathematical proof would the many consequences of the conjecture be valid. This is nicely depicted in a book 'Prime Obsession' by John Derbyshire.

Coming back to what you said earlier about model and theory, this will be for the theory part, more for like the proof in the understanding?
It's the part that I'm interested in, the proof, yes.

So what is a good model? We mentioned this a little bit earlier.
Well, as we said before, a good model has to be intuitive, it has to be pretty, it should be possible to encapsulate it simply, although understanding it may be extremely complicated. Again I can quote something from mathematics: Fermat's Theorem is a very simple mathematical theorem that any high school student can understand, and yet it took 350 years for mathematicians to show that it was right.

So this was really the first part, on the modeling side. And now the second part is about risk and uncertainty, which may also be very different in your field and your experience. How would you define uncertainty?
Well**,** I think there are different types of uncertainty. First, when you're dealing with human beings you can always make errors, everybody at some point makes mistakes, even computers make mistakes as well. So you have built-in the fact that certainty isn't guaranteed in our life. On the other hand, mathematics probably is more certain than anything else in science. So we believe that the degree of certainty - you have to talk about degree of certainty and degree of risk – and the risk is

rather low if you check something very carefully, and multiple people check it, that it's not correct. But there's another type of uncertainty, and that's the uncertainty of what the human mind can do, or what the mathematical theory can do, and we never know, for instance, if you have a model or a hypothesis, whether you can really show if it's either right or wrong. And that's another type of uncertainty: you might spend your whole lifetime trying to show that a certain model is right, and never achieve it; and you don't know if you even can achieve it. So that's another type of risk, which you have to manage, mainly by having good intuition about choosing good things to work on.

Do you make a distinction between uncertainty and risk? Or not really?
No not really. While they're certainly different, they're so tied up together that their practical consequences can be the same. Offhand I haven't thought that one through.
That's fine. For economists, for instance, there is a huge difference, but it really depends on the field; and I guess in your field, because as you said, mathematics is somehow really clean in terms of risk associated with it, so…
Risk, I think, is involved in going off in the wrong direction, in a tangent. And there it's not a question of uncertainty but the risk that you're not going in a fruitful direction.

And so it seems, it's interesting – the only expert I can relate at this stage is the cosmologist I interviewed, because for him the huge, the progress in his field was related to the development of tools, observation tools and things like this. So it seems that in your field, the development of mathematics and mathematical areas have a huge impact on what can be done. If it's not mature enough then people will not be able to make progress during their lifetime, so we need to wait several generations.
Well in my generation something which happened which is very important and changed the way that everybody thought about these subjects is that it used to be, maybe 60 years ago, that people were extremely specialized, and you developed A, B, C, D or E. And then people started to discover relationships between A and B, B and C, D and E, and now more or less everything seems to be related. And that makes it very much more interesting, and also very much more difficult, because instead of specializing and becoming an expert, you have to have some knowledge across many different fields in order to be able to do that. That makes the work so much more difficult. Not only in mathematics has this connection come about, but between mathematics and Nature, between mathematics and physics. So within physics, in fields of physics that people thought were separate, as I mentioned before - like statistical physics, quantum mechanics, classical mechanics – they all now can be related by certain ideas, and in order to see these relations you also need different aspects of mathematics that seemed before to apply one to one, another to another, but now they are all connected. So this makes everything exciting. That's a big change that's happened during my lifetime.

Did you find it difficult to interact with people at the beginning who had a different expertise, still in physics, but different fields of physics? Since now things are more inter-disciplinary within physics – do you have any communication problems, or any difficulties at this stage?
Yes there are major difficulties, and the difficulty comes because now there are so many more scientists and so many more mathematicians, so each individual scientist or mathematician is surrounded by a cloud of colleagues and students, and that makes it very much more difficult to interact with people in different fields.

I was more thinking in terms of language? Maybe what people mean by certain words, or by certain concepts?
Well sociologically people become isolated in certain groups, and so even though interaction should take place more and more, however from a practical side it becomes more difficult.

The last question, which is quite a personal question: what is the experience or result in your field (so it can be one of your own but it can be something also more general) which has had the most significant impact for you, and why?
Well I think it is what I just explained – that everything has become interconnected, and this has become very important for me, because I was involved in making some of the connections, but I could see everywhere in the field this was happening, and that's very exciting. This is why to be successful one needs to be a student one's entire life. One must always be learning something new!

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In the 1970s Jaffe was active in the foundation of the International Association of Mathematical Physics and served for six years as its President (1990-1996). He co-founded a sequence of summer schools on mathematical physics that met in Cargèse, Corsica from 1976 to 1996. He acted as Chief Editor of *Communications in Mathematical Physics* for 22 years. Jaffe was Chair of the Harvard University department of mathematics, after which he became President of the American Mathematical Society. In 1998, Jaffe guided the conception of the Clay Mathematics Institute, then serving as its President and designing all its programs, including the "Millennium Problems in Mathematics."

Jaffe received the Dannie Heineman Prize for Mathematical Physics from the American Physical Society and the American Institute of Physics. and the Prize for Mathematical and Physical Sciences from the New York Academy of Sciences. He is a fellow of the American Academy of Arts and Sciences, a member of the U.S. National Academy of Sciences, and an honorary member of the Royal Irish Academy. He also has had a life-long interest in classical music.